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SOME APPROXIMATE METHODS OF SOLVING INTEGRAL EQUATIONS OF MIXED PROBLEMS*

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Two algorithms are developed for investigating an integral equation (IE) that arises in the study of mixed problems of the mechanics of continuous media with boundary conditions specified on a circle. The first is a generalization of the orthogonal function method and relies on the approximate construction of the sequence of eigenvalues and a corresponding system of eigenfunctions of the integral operator of the original problem. It is shown that this approach is effective for any values of a certain non-dimensional parameter $\lambda \in (0, \infty)$ of geometrical or physical origin, occurring in the kernel of the integral equation. The second method is applicable for small λ values and is based on Koiter's idea of approximate factorization. Its advantage is its greater accuracy compared with the previously used method, which involved approximation of the kernel of the integral equation. As an example, we present the solution of an axisymmetric problem: the impression of a stamp into an elastic half-space reinforced at the boundary by a thin cover.

1. It is well-known /1, 2/ that a wide range of three-dimensional mixed problems in the mechanics of continuous media and mathematical physics, in which the boundary conditions are specified on a circle, reduce to a consideration of an integral equation of the form

$$\int_0^1 \varphi(\rho)\rho k\left(\frac{\rho}{\lambda}, \frac{r}{\lambda}\right) d\rho = \lambda f(r) \quad (0 \leqslant r \leqslant 1) \tag{1.1}$$

$$k(\beta, \alpha) = \int_0^\infty K(u) u J_0(u\beta) J_0(u\alpha)\, du \quad \left(\beta = \frac{\rho}{\lambda},\ \alpha = \frac{r}{\lambda}\right) \tag{1.2}$$

The kernel symbol $K(\zeta)$ has the following properties /3/: 1) it is an even function, $K(u) > 0$ $(|u| < \infty)$; 2) in the plane of the complex variable $\zeta = u + iv$ the function $K(\zeta)$ is regular in the strip $|u| < \infty$, $|v| < \delta$ and continuous on the real axis except for the point $\zeta = 0$; 3) $K(\zeta)$ satisfies the following asymptotic formulae on the real axis:

$$K(u) \sim |u|^{-1} \ (|u| \to \infty), \quad K(u) \sim B\,|u|^{-1} \quad (u \to 0) \tag{1.3}$$

Using (1.3) and the values of the integral /4/

$$\int_0^\infty J_0(u\beta) J_0(u\alpha)\, du = \frac{2}{\pi(\beta+\alpha)} \mathbf{K}(e), \quad e = \frac{2\sqrt{\beta\alpha}}{\beta+\alpha} \tag{1.4}$$

where $\mathbf{K}(e)$ is the complete elliptic integral of the first kind, we infer from (1.2) that

$$k(\beta, \alpha) \sim \frac{2}{\pi(\beta+\alpha)} \mathbf{K}(e) \quad (|\beta - \alpha| \to 0) \tag{1.5}$$
$$k(\beta, \alpha) \sim \frac{2B}{\pi(\beta+\alpha)} \mathbf{K}(e) \quad (|\beta - \alpha| \to \infty)$$

It should be borne in mind that the integral Eqs.(1.1), (1.2) can be reduced by the "method of transforming operators" /5/ to an equivalent equation of the second kind:

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$$\pi\lambda\psi(x)+\int_{-1}^{1}\psi(\xi)\,k\left(\frac{\xi-x}{\lambda}\right)d\xi=\pi\lambda g(x)\quad(|x|\leqslant 1) \tag{1.6}$$

$$k(t)=\int_0^\infty [L(u)-1]\cos ut\,du,\quad K(u)\,|u|=L(u) \tag{1.7}$$

Here

$$\varphi(r)=\frac{2}{\pi}\left\{\frac{\psi(1)}{\sqrt{1-r^2}}-\int_r^1\frac{\psi'(\rho)\,d\rho}{\sqrt{\rho^2-r^2}}\right\} \tag{1.8}$$

$$g(x)=f(0)+|x|\int_0^{|x|}\frac{f'(\rho)\,d\rho}{\sqrt{x^2-\rho^2}} \tag{1.9}$$

Apart from the above properties of the kernel symbol $K(\zeta)$ in (1.2) or (1.7), we assume that the following formulae hold:

$$L(u)=1+\sum_{n=1}^{N}c_n u^{-n}+O(u^{-N-1})\quad(u\to\infty) \tag{1.10}$$

$$\left|L(u)-1-\frac{c_1}{u}-\frac{c_2}{u^2}\right|\leqslant\frac{a}{u^2(u+b)}\quad(0\leqslant u\leqslant\infty;\ a,b>0) \tag{1.11}$$

We then have the following representation /3/:

$$k(t)=R_0-c_1\ln|t|-{}^1/_2\pi c_2|t|+l(t)\quad(R_0=\text{const})$$

where $l(t)$ is a function whose first derivative satisfies a Hölder condition for $|t|\leqslant R<\infty$, with exponent $1-\varepsilon>0$ $(\varepsilon>0)$, i.e., $l(t)\in H_1^{1-\varepsilon}(-R,R)$. Hence it follows that (1.6) is a Fredholm integral equation.

We assert that if $f'(r)\in L_p(\Omega)$ $(p>2)$ ($L_p(\Omega)$ is the space of functions summable in the disc Ω: $0\leqslant r\leqslant 1$ to power p), then $g(x)\in H_0^1(-1,1)$. To prove this we clearly have to show that the integral (1.9) is bounded. We use the Holder inequality /6/

$$\left|\int_0^{|x|}\frac{f'(\rho)\,d\rho}{\sqrt{x^2-\rho^2}}\right|\leqslant\left(\int_0^{|x|}\rho|f'(\rho)|^p d\rho\right)^{1/p}\left[\int_0^{|x|}\left(\frac{\rho^{-1/p}}{\sqrt{x^2-\rho^2}}\right)^q d\rho\right]^{1/q}$$
$$(1/p+1/q=1)$$

from which it follows that $p>2$ and so $g(x)\in H_0^1(-1,1)$.

It can be proved that if the right-hand side of (1.6), (1.7) is such that $g(x)\in H_0^1(-1,1)$, then the solution of this integral equation has the following properties:

$$\psi(x)\in H_0^1(-1+\varepsilon,\ 1-\varepsilon),\quad\psi(x)\in H_0^{1-0}(-1,\ -1+\varepsilon) \tag{1.12}$$
$$\psi(x)\in H_0^{1-0}(1-\varepsilon,\ 1)$$

(ε is a positive number as small as desired).

Indeed, consider the integral

$$F(x)=-\int_{-1}^{1}\psi(\xi)\left[c_1\ln\left|\frac{\xi-x}{\lambda}\right|+\frac{\pi c_2}{2}\left|\frac{\xi-x}{\lambda}\right|\right]d\xi \tag{1.13}$$

which is the principal part of the kernel (1.11) of Eq.(1.6). Since $g(x)\in H_0^1(-1,1)$, and since (1.6) is a Fredholm equation of the second kind, the latter is solvable at least in the space of continuous functions $C(-1,1)$ for almost all values of the parameter $\lambda\in(0,\infty)$ (it can be shown that the integral Eqs.(1.6) and (1.7) are solvable for all $\lambda\in(0,\infty)$).

Differentiating both sides of (1.13) with respect to x and remembering that $\psi(x)\in C(-1,1)$, we conclude /7/ that the function $F(x)$ satisfies conditions (1.12). Since the right-hand side of Eq.(1.6) is a function of class $H_0^1(-1,1)$, it follows, comparing the properties of $g(x)$ and $F(x)$, that (1.12) is true.

We will now examine the structure of the solution of the original integral Eqs.(1.1) and (1.2). We assert that if $f'(r)\in L_p(\Omega)$, then the function $\varphi(r)$ as defined in (1.8) may be expressed as

$$\varphi(r)=\omega(r)/\sqrt{1-r^2},\quad\omega(r)\in C(\Omega) \tag{1.14}$$

To prove this, we study the properties of the integral in (1.8):

$$I(r)=\sqrt{1-r^2}\int_r^1 \frac{\psi'(\rho)\,d\rho}{\sqrt{\rho^2-r^2}}=\psi(1)-\psi(r)+ \tag{1.15}$$

$$\sqrt{1-r^2}\int_r^1 \frac{\Psi(\rho,r)}{\sqrt{\rho-r}}\,d\rho,\quad \Psi(\rho,r)=\frac{\rho[\psi(\rho)-\psi(r)]}{(\rho^2-r^2)\sqrt{\rho+r}}$$

Note that in view of the properties of $\psi(r)$ established above, the function $\Psi(\rho, r)$ in (1.15) exhibits the following asymptotic behaviour as $\rho\to r$:

$$\Psi(\rho,\ r)=\Psi(r,\ r)+o(\rho-r),\quad \Psi(r,\ r)\in C(\Omega^*) \tag{1.16}$$

$$\Psi(\rho,\ r)\sim(\rho-r)^{-\varepsilon}\quad (\rho,\ r\in\Omega\setminus\Omega^*)$$

where Ω^* is a disc of radius $1-\varepsilon$. Substituting (1.16) into the last integral of (1.15), we obtain $I(r)\sim\Psi(r,\ r)$ $(r\in\Omega^*)$, $I(r)\sim(1-r)^{1-\varepsilon}$ $(r\in\Omega\setminus\Omega^*)$, whence it follows that $I(r)\in C(\Omega)$, and so also $\omega(r)\in C(\Omega)$. We have thus proved the following

Theorem. If $f'(r)\in L_p(\Omega)$ $(p>2)$, then the integral Eq.(1.1) with kernel (1.2) and symbol $K(u)$ satisfying conditions 1–3 is uniquely solvable in $L_q(\Omega)$ $(1<q<2)$ and its solution $\varphi(r)$ has the structure (1.14). Moreover, the solution satisfies the well-posedness relations

$$\|\varphi\|_{L_q}\leqslant\theta_1(\lambda)\|f'\|_{L_p},\quad \|\omega\|_C\leqslant\theta_2(\lambda)\|f'\|_{L_p}$$

where $\theta_1(\lambda)$ and $\theta_2(\lambda)$ are bounded constants for any fixed λ.

2. Before proceeding to construct a solution of the original integral Eqs.(1.1) and (1.2), we consider the properties of its kernel in more detail. Suppose that the kernel symbol satisfies conditions (1.3) and behaves asymptotically at infinity as in (1.10). Then, in view of the integrals (1.4) and /4/,

$$\int_0^\infty \frac{J_0(\beta u)J_0(\alpha u)-e^{-u/2}}{u}\,du=-\ln\frac{\beta+\alpha+|\beta-\alpha|}{2}$$

we represent $k(\beta,\alpha)$ (see (1.2)) as

$$k(\beta,\alpha)=\frac{2}{\pi(\beta+\alpha)}\mathbf{K}(e)-c_1\ln\frac{\beta+\alpha+|\beta-\alpha|}{2}+m(\beta,\alpha) \tag{2.1}$$

$$m(\beta,\alpha)=\int_0^\infty\left\{\left[L(u)-1-\frac{c_1}{u}\right]J_0(\beta u)J_0(\alpha u)+\frac{c_1}{u}e^{-u/2}\right\}du$$

where, as can be shown, $m(\beta,\alpha)$ is an at least continuously differentiable function of its arguments on the square $0\leqslant\alpha,\ \beta<\infty$.

Now, in accordance with the theorem, we consider the Hilbert space $L_2^{1/2}(\Omega)$ of functions which are square summable in the disc Ω: $0\leqslant r\leqslant 1$ with weight $(1-r^2)^{-1/2}$ and look for the solution $\varphi(r)$ of equations (1.1) and (1.2) in the form of (1.4), where

$$\omega(r)=\sum_{n=1}^\infty d_n\omega_n(r) \tag{2.2}$$

The functions $\omega_n(r)$ in (2.2) are eigenfunctions of the operator

$$\mathbf{A}\omega=\frac{1}{\lambda}\int_0^1\frac{\omega(\rho)\rho}{\sqrt{1-\rho^2}}k\left(\frac{\rho}{\lambda},\frac{r}{\lambda}\right)d\rho \tag{2.3}$$

i.e., non-trivial solutions of the homogeneous equation

$$\mathbf{A}\omega_n=\mu_n\omega_n\quad(0\leqslant r\leqslant 1) \tag{2.4}$$

Note that in accordance with the representation (2.1) and the symmetry of the kernel (1.2) in Ω, the system of eigenfunctions $\{\omega_n(r)\}$ is orthonormal and complete in $L_2^{1/2}(\Omega)$, and the series (2.2) converges in the norm of the space $L_2^{1/2}(\Omega)$; moreover, $\{d_n\}\in l_2$.

We now expand the function $f(r)$ in a Fourier series, uniformly convergent in Ω_* in terms of the system $\{\omega_n(r)\}$:

$$f(r) = \sum_{n=1}^{\infty} f_n \omega_n(r), \quad f_n = \int_0^1 \frac{rf(r)\,\omega_n(r)}{\sqrt{1-r^2}}\,dr \tag{2.5}$$

Substituting (2.2) and (2.5) into (1.1), using (2.4) and equating coefficients of corresponding eigenvalues on the right and left of the resulting equation, we obtain

$$d_n = f_n \mu_n^{-1} \quad (n = 1, 2, 3, \ldots) \tag{2.6}$$

and so we finally write the solution of Eqs.(1.1) and (1.2) in the form (1.14), (2.2), (2.6), on the assumption that the real eigenvalues $\{\mu_n\}$ and corresponding eigenfunctions $\{\omega_n(r)\}$ of the operator $\mathbf{A}\omega$ are given.

That (2.6) is legitimate follows from the fact that the operator (2.3) is positive definite.

Indeed, consider the scalar product

$$(\mathbf{A}\omega, \omega)_{L_2^{1/2}} = \int_0^\infty L(u)\,H^2\left(\frac{u}{\lambda}\right)du \tag{2.7}$$

$$H\left(\frac{u}{\lambda}\right) = \frac{1}{\sqrt{\lambda}} \int_0^1 \frac{r\omega(r)}{\sqrt{1-r^2}}\,J_0\left(\frac{ur}{\lambda}\right)dr$$

Thanks to the asymptotic properties of $L(u) = K(u)\,|u|$ (1.3) and the Parseval equality for the Hankel transform, the integral on the right of the first relation in (2.7) is convergent.

Since $L(u) > 0$ on the real axis $0 \leqslant u < \infty$, it follows that

$$(\mathbf{A}\omega, \omega)_{L_2^{1/2}} = \gamma(\omega, \omega)_{L_2^{1/2}} > 0 \quad (\gamma = \mathrm{const})$$

and hence the operator (2.3) is positive definite. Then /6/ $0 < \gamma < \ldots < \mu_n < \ldots < \mu_2 < \mu_1$, and this justifies (2.6).

We will now determine the eigenfunctions of the operator $\mathbf{A}\omega$ or, what is the same, the solution of the homogeneous integral Eq.(2.4). To that end we can use, for example, the Ritz method /8, 9/. As a sequence of coordinate elements we take the system of Legendre polynomials $\{P_{2n}^*(\sqrt{1-r^2})\}$:

$$\omega_n^N(r) = \sum_{m=0}^{N} b_m^{(n)} P_{2m}^*(\sqrt{1-r^2}), \quad P_{2m}^*(\sqrt{1-r^2}) = \sqrt{4m+1}\,P_{2m}(\sqrt{1-r^2}) \tag{2.8}$$

It is well-known /6/ that this system is a basis in $L_2^{1/2}(\Omega)$. We substitute (2.8) into (2.4) and then take the scalar product of both sides of the resulting expression with $P_{2k}^*(\sqrt{1-r^2})$. In view of the orthonormality of the polynomials $P_{2m}^*(\sqrt{1-r^2})$, we obtain

$$\sum_{m=0}^{N} c_{km} b_m^{(n)} = \mu_n b_k^{(n)} \quad (k = 0, 1, \ldots, N;\ n \geqslant 1) \tag{2.9}$$

$$c_{km} = (\mathbf{A}P_{2k}^*, P_{2m}^*)_{L_2^{1/2}} \tag{2.10}$$

Using the values of the integral /4/ $(u \geqslant 0)$

$$\int_0^1 J_0(ur)\,P_{2n}(\sqrt{1-r^2})\,\frac{r\,dr}{\sqrt{1-r^2}} = \sqrt{\frac{\pi}{2u}}\,\alpha_n J_{2n+1/2}(u)$$

$$\alpha_n = (2n-1)!!\,/(2n)!!$$

we can write the coefficients c_{km} as

$$c_{km} = {}^1/_2\pi\sqrt{(4k+1)(4m+1)}\,\alpha_k\alpha_m \int_0^\infty K(u)\,J_{2k+1/2}\left(\frac{u}{\lambda}\right)J_{2m+1/2}\left(\frac{u}{\lambda}\right)du \tag{2.11}$$

For system (2.9) and (2.10) to have a non-trivial solution, its determinant must vanish. Equating the determinant to zero, we obtain an equation for the first N eigenvalues μ_n of the operator (2.3). Once μ_n have been determined, we can find $b_m^{(n)}$, expressing them in terms of $b_0^{(n)}$:

$$b_m^{(n)} = b_0^{(n)} h_m^{(n)} \quad (h_0^{(n)} = 1) \tag{2.12}$$

In view of (2.8), we obtain

$$\omega_n^N(r) = b_0^{(n)} \psi_n^N(r), \quad \psi_n^N(r) = \sum_{m=0}^{N} h_m^{(n)} P_{2m}^*(\sqrt{1-r^2}) \tag{2.13}$$

The constants $b_0^{(n)}$ $(n \geqslant 1)$ in (2.12), (2.13) are determined by normalizing the eigenfunctions $\omega_n^N(r)$, i.e.,

$$\int_0^1 \frac{r[\omega_n^N(r)]^2}{\sqrt{1-r^2}} dr = [b_0^{(n)}]^2 \sum_{m=0}^{N} [h_m^{(n)}]^2 = 1 \tag{2.14}$$

After finding $b_0^{(n)}$ from (2.14), we determine approximations of the required eigenfunctions of the operator $\mathbf{A}\omega$. In so doing we use the fact that, thanks to the previously indicated properties of the operator (2.3), the Ritz process for the integral Eq.(2.4) will converge /8/, i.e., $\omega_n^N(r) \to \omega_n(r)$ $(N \to \infty)$. In addition, by formulae (1.5) and the spectral formula /10/

$$\int_0^1 \frac{\rho P_{2m}(\sqrt{1-\rho^2})}{\sqrt{1-\rho^2}} \mathrm{K}\left(\frac{2\sqrt{\rho r}}{\rho + r}\right) \frac{d\rho}{\rho + r} = \frac{\pi^2}{4} \alpha_m{}^2 P_{2m}(\sqrt{1-r^2})$$

the coefficients c_{km} given by (2.11) exhibit the following asymptotic behaviour:

$$c_{km} \sim {}^1/{}_2\pi\alpha_k{}^2\delta_{km} \quad (\lambda \to \infty), \quad c_{km} \sim {}^1/{}_2\pi B\alpha_k{}^2\delta_{km} \quad (\lambda \to 0)$$

(δ_{km} is the Kronecker delta), which implies that as $\lambda \to \infty$ or $\lambda \to 0$ the matrix of the system (2.9) becomes diagonal.

3. We will now present an algorithm for investigating integral Eqs.(1.1) and (1.2), which is effective for small λ values. Using a formula due to Krein /11/, we consider Eqs. (1.6) and (1.7):

$$\int_{-1}^{1} \psi(\xi)\, k\left(\frac{\xi - x}{\lambda}\right) d\xi = \pi\lambda g \quad (|x| \leqslant 1, \ g = f(0) = \mathrm{const}) \tag{3.1}$$

$$k(t) = \int_0^\infty L(u) \cos ut\, du \tag{3.2}$$

With an eye to solving integral Eq.(3.1) for $\lambda \ll 1$ by Koiter's approximate factorization method /12/, we use formulae (1.3), (1.10) to approximate the kernel symbol $K(\zeta)\,|\,\zeta\,| = L(\zeta)$ as follows:

$$L(\zeta) \approx L_*(\zeta) = \frac{\zeta^2 + h_3{}^2}{\zeta^2 + h_4{}^2} \exp\left(\frac{h_1\sqrt{\zeta^2 + \varepsilon^2}}{\zeta^2 + h_2{}^2}\right) \quad (\varepsilon \to 0, \ B = h_3{}^2 h_4^{-2}) \tag{3.3}$$

We will seek the principal term of the asymptotic expansion of the solution to an integral Eq.(3.1) with kernel (3.2), (3.3) in the following form /3/:

$$\psi(x) = g\left[\omega\left(\frac{1+x}{\lambda}\right) + \omega\left(\frac{1-x}{\lambda}\right) - \frac{1}{B}\right] \tag{3.4}$$

where $\omega(s)$ satisfies the Wiener-Hopf integral equation, whose solution is /3/

$$\omega(s) = \frac{1}{2\pi i L_-(0)} \int_\Gamma \frac{e^{-i\zeta s}}{\zeta L_+(\zeta)} d\zeta \quad (0 \leqslant s < \infty) \tag{3.5}$$

Here the contour Γ is a straight line just above the real axis in the plane of the complex variable $\zeta = u + iv$, and $L_*(\zeta) = L_+(\zeta) L_-(\zeta)$, where the functions $L_+(\zeta)$ and $L_-(\zeta)$ are regular in the half-planes $\mathrm{Im}\,\zeta > -\varepsilon$, $\mathrm{Im}\,\zeta < \varepsilon$, respectively, have no zeros there and can be written as /13/

$$L_\pm(\zeta) = \frac{\zeta + ih_3}{\zeta \pm ih_4} \exp [h_1 \mu_\pm(\zeta)] \tag{3.6}$$

$$\mu_\pm(\zeta) = \frac{-i\zeta\, [i\zeta\nu_\pm(\zeta) \pm h_2\nu_\pm(\pm ih_2)]}{\zeta^2 + h_2^2}$$

$$\nu_\pm(\zeta) = \pm \frac{i}{\pi\sqrt{\zeta^2 + \varepsilon^2}} \ln \frac{\zeta + \sqrt{\zeta^2+\varepsilon^2}}{\pm i\varepsilon}$$

It follows from (3.6) that

$$L_-(0) = L_+(0) = \sqrt{B} \tag{3.7}$$

We will now substitute (3.7) into (3.5) and for convenience express the latter in terms of the Laplace-Carson transform rather than the integral Fourier transform, by substituting $\zeta = ip$. By (3.6), we obtain

$$\omega(s) = \frac{1}{2\pi\sqrt{B}\, i} \int_L \frac{\Omega(p)}{p} e^{ps}\, dp \tag{3.8}$$

where L is a straight line just to the right of the imaginary axis in the plane of the complex variable p, and we have used the notation

$$\Omega(p) = \frac{p + h_4}{p + h_3} \exp[-h_1\mu(p)], \quad \mu(p) = \frac{p\,[p\nu(p) - h_2\nu(h_2)]}{p^2 - h_2^2} \tag{3.9}$$

$$\nu(p) = \frac{1}{\pi\sqrt{p^2-\varepsilon^2}} \ln \frac{p + \sqrt{p^2-\varepsilon^2}}{\varepsilon}, \quad \nu(h_2) = \nu_+(ih_2)$$

Now, transforming the expression for $\mu(p)$ with the help of the third formula of (3.9), we find that

$$\mu(p)\, [\pi(p^2 - h_2^2)]^{-1} p \ln(ph_2^{-1}) \tag{3.10}$$

Note that the function $\exp[-h_1\mu(p)]$ may be approximated in the half-plane $\operatorname{Re} p > 0$ to a high degree of accuracy by the expression /14/

$$\exp[-h_1\mu(p)] \approx 1 - h_1\mu(p) \tag{3.11}$$

The error in this approximation over the positive real axis, for example /13/, is at most 1% for the h_1 and h_2 values specified in Sect.4.

Substituting (3.9)-(3.11) into (3.8) and using tables /15/, we obtain ($\operatorname{Ei}(x)$ is the integral error function)

$$\omega(s) = \frac{1}{B} + \frac{1}{\sqrt{B}}\left(1 - \frac{1}{\sqrt{B}}\right) e^{-h_3 s} + J(s) \tag{3.12}$$

$$J(s) = \frac{h_1}{\pi\sqrt{B}\,(h_3 - h_2)} \int_0^s e^{h_2\tau} \operatorname{Ei}(-h_2\tau)\, [(h_3 - h_4)\, e^{-h_3(s-\tau)} +$$

$$(h_4 - h_2)\, e^{-h_2(s-\tau)}]\, d\tau$$

Thus the solution of the integral Eqs.(1.1) and (1.2) for $\lambda \ll 1$ can be written as (1.8), (3.4) and (3.12).

4. As an example, let us consider an axisymmetric contact problem, in which a stamp of circular plan $(0 \leqslant r \leqslant a)$ and flat base is impressed without friction into an elastic (G_2, ν_2) half-space whose surface is reinforced by a cover plate /16, 17/:

$$2G_1 h \Delta u_\pm = -(1 - \nu_1)(\tau_+ - \tau_-) - 0.5\nu_1 h\, (\sigma_+' + \sigma_-') \tag{4.1}$$

$$\sigma_+ - \sigma_- = -\frac{h}{2r} [r(\tau_+ + \tau_-)]', \quad \Delta = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}$$

Here G_1, ν_1 are the elastic constants of the cover material, h its thickness, and $\sigma_\pm(r)$ and $\tau_\pm(r)$ the normal and tangential stresses acting on the upper and lower faces of the cover.

The problem is to determine the distribution law of the contact pressures $q(r)$, as well as the relationship between the depression δ of the base and the force exerted on the stamp

$$P = 2\pi \int_0^a rq(r)\, dr \tag{4.2}$$

Using a Hankel integral tansformation with respect to r, one can reduce this problem to determining $q(r)$ from an integral equation of the first kind. Introducing non-dimensional variables $\rho^* = \rho a^{-1}$, $r^* = ra^{-1}$ and the notation $\varphi(r^*) = q(r)\theta_2^{-1}$, $N_0 = P(a\theta_2)^{-1}$, $f(r^*) = f(1-\varepsilon_2^2)^{-1} = g$, $f = \delta a^{-1}$, $\varepsilon_2 = 0.5(1-2\nu_2)(1-\nu_2)^{-1}$, $\theta_j = G_j(1-\nu_j)^{-1}$, $\lambda = 2hna^{-1}$, $n = \theta_1\theta_2^{-1}$ (the asterisk will be omitted henceforth), we can express the integral equation in question in the form of (1.1), with the function $k(\beta, \alpha)$ given by formula (1.2) in which $K(u)$ is

$$K(u) = \frac{u + (1-\varepsilon_2^2)^{-1}}{u(u+1)}, \quad B = \frac{1}{1-\varepsilon_2^2} \tag{4.3}$$

The following observation is important. Asymptotic analysis of the contact problem of the impression of a stamp into a two-layer base /18/ has established that if the relative thickness of the cover ha^{-1} is small but its relative stiffness n is large, where $n^{-1} = O(ha^{-1})$ $(ha^{-1} \to 0)$, then its physico-mechanical properties may be described up to $O(ha^{-1})$ terms by the equations of the cover (4.1).

In order to obtain numerical results for the original problem for small λ values, i.e., to use the formulae of Sect.3, we must specify the values of the constants h_j $(j = 1, 2, 3, 4)$ in (3.3). For example, for $\nu_2 = 0.3$ in (4.3) we set $h_1 = -0.09796$, $h_2 = 1.0954$, $h_3 = 3.9044$, $h_4 = 3.7417$. In that case the error incurred by using (3.3) to approximate the kernel symbol $K(u)$ of (4.3) is at most 1% over the whole real axis.

λ	$r=0$	0.2	0.4	0.6	0.8	0.95	$N_0 f^{-1}$
0.25	0.594 0.591	0.599 0.597	0.633 0.635	0.735 0.737	0.982 0.981	1.909 1.911	3.74 3.75
0.5	0.673 0.677	0.633 0.636	0.607 0.611	0.741 0.745	1.018 1.020	1.864 1.888	3.77 3.81
1	0.626 0.624	0.617 0.615	0.637 0.639	0.757 0.759	1.003 1.007	1.915 1.920	3.82 3.85
2	0.647 0.653	0.630 0.637	0.642 0.643	0.771 0.778	1.015 1.012	1.917 1.925	3.87 3.89
4	0.656	0.640	0.651	0.781	1.027	1.934	3.91

The table shows the values of the contact pressures $\varphi(r)f^{-1}$ and force $N_0 f^{-1}$ (4.2) on the stamp, calculated by the methods of Sect.2 (the first row) and Sect.3 (the second row). The solution (2.2) was considered up to the first seven eigenfunctions of $A\omega$, and the error of the approximate solution does not exceed 1.5% for any parameter values $\lambda \in (0, \infty)$. It is obvious that as the relative stiffness or relative thickness of the cover increases, there is an increase in the contact pressures and the force on the stamp, the latter varying within the limits $N_0 f^{-1} = 3.67$ (as $\lambda \to 0$) and $N_0 f^{-1} = 4$ (as $\lambda \to \infty$) /19/. Hence it follows that failure to allow for the influence of a thin reinforcing cover may lead to an error in determining the contact stiffness of a composite base, amounting to 9% for $\nu_2 = 0.3$.

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A NEW APPROACH TO THE ASYMPTOTIC INTEGRATION OF THE EQUATIONS OF SHALLOW CONVEX SHELL THEORY IN THE POST-CRITICAL STAGE*

A.YU. EVKIN

A method is proposed for the asymptotic integration of the non-linear equations of shallow elastic shell theory on the basis of a new definition of the small parameter that is selected to be proportional to the ratio between the shell thickness and the amplitude of its deflection. This parameter is actually small if the shell is in the post-critical stage, i.e., its deflections are large. An asymptotic expansion of the solution of the shell equilibrium equations in the parameter mentioned is carried out. It is established that the first two approximations result in the geometric theory of shell stability formulated by Pogorelov /1/. By comparing the asymptotic and numerical solutions /2/ found for a spherical shell under axisymmetric deformation, satisfactory accuracy of the proposed method is obtained for fairly large deflection. The well-known Koiter approach is used in the small-deflection domain. The two asymptotic expansions, one of which is suitable for small deflections and the other for large, are merged using the Padé approximation.

Despite the efficiency of the well-known asymptotic method (/3-5/, etc.) in non-linear shell theory, the singularities of the non-linear equations describing the behaviour of the shell for deflections substantially exceeding its thickness are not used therein. The significant post-critical shell deformations are described well in a number of cases by the Pogorelov /1/ geometric theory which is, however, phenomenological in nature. The investigations in /3-7/ are devoted to proving the geometrical method. The paper by Lesnichaya /7/ should be noted, in which the ratio between the shell thickness and the characteristic dimension of the domain of the post-critical dents is utilized as the small parameter in a study of the axisymmetric deformation of a closed sphere under uniform external pressure. Relationships of the geometrical theory are obtained as the fundamental approximation. However, the connection

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